

Approximation by convolutions with probability densities and applications

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Abstract

Abstract. The purpose of this paper is to introduce several new convolution operators, generated by some known probability densities. Approximation results of these convolution operators are proved. Also, by using the inverse Fourier transform and taking inverse steps (in the analogues of the classical procedures used for, e.g., the heat or Laplace equations), we deduce the initial value problems satisfied by the new convolution integrals. The paper ends with the study of a convolution singular improper integral based on a special, unbounded density called horseshoe density.

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1 Introduction

It is well known the fact that the classical Gauss-Weierstrass, Poisson-Cauchy and Picard convolution singular integrals are based on convolutions with the normal density function e^{-x^2} , Cauchy density function $\frac{1}{1+x^2}$ and Laplace density function $e^{-|x|}$, respectively. Their approximation properties are studied, for example, in [2], [8]. Also, by using the Fourier transform method, it is known that the solutions of the initial value problems for the heat equation and Laplace equation are exactly the Gauss-Weierstrass and Poisson-Cauchy convolution singular integrals, respectively, see, e.g., [9], p. 23.

The main aim of the present paper is somehow inverse : introducing convolution singular integrals based on some known probability densities, after obtaining their approximation properties, we use the inverse Fourier transform in order to find the partial differential equations (initial value problems) satisfied by these integrals. The paper ends with the study

of a convolution singular improper integral based on a unbounded density called horseshoe density.

2 Definitions of Convolution Operators

In this section we introduce several convolution operators, based on some well-known densities of probability.

If $d(t, x)$ with $t > 0$ and $x \in \mathbb{R}$ is a probability density, that is $\int_{-\infty}^{+\infty} d(t, x) dx = 1$, then our definitions are based on the general known formula

$$O_t(f)(x) = d(t, \cdot) * f(\cdot) = \int_{-\infty}^{+\infty} f(u) \cdot d(t, x - u) du = \int_{-\infty}^{+\infty} f(x - v) \cdot d(t, v) dv. \quad (1)$$

Definition 2.1 (i) For the Maxwell-Boltzmann type probability density (see, e.g., [11])

$$d(t, x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{x^2 e^{-x^2/(2t^2)}}{t^3}, x \in \mathbb{R}, t > 0$$

and $f : \mathbb{R} \rightarrow \mathbb{R}$, we can formally define the Maxwell-Boltzmann convolution operator

$$S_t(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x - v) \cdot \frac{v^2 e^{-v^2/(2t^2)}}{t^3} dv, t > 0, x \in \mathbb{R}. \quad (2)$$

(ii) For the Laplace type probability density (see, e.g., [7])

$$d(t, x) = \frac{1}{2t} e^{-|x|/t}, t > 0, x \in \mathbb{R}$$

and $f : \mathbb{R} \rightarrow \mathbb{R}$, we can formally define the classical Picard convolution operator

$$P_t(f)(x) = \frac{1}{2t} \int_{-\infty}^{+\infty} f(x - v) \cdot e^{-|v|/t} dv, t > 0, x \in \mathbb{R}. \quad (3)$$

(iii) For the exponential probability density (see, e.g., [7], [1], [10])

$$d(t, x) = \frac{t e^{-t|x|}}{2}, x \in \mathbb{R}, t > 0$$

and $f : \mathbb{R} \rightarrow \mathbb{R}$, we can formally define the exponential convolution operator

$$E_t(f)(x) = \int_{-\infty}^{+\infty} f(x - v) \cdot \frac{t e^{-t|v|}}{2} dv, t > 0, x \in \mathbb{R}. \quad (4)$$

(iv) For any $n \in \mathbb{N}$, $P_t(f)(x)$ can be generalized to the so called Jackson type generalization of the Picard singular integral defined by (see, e.g., [8])

$$P_{n,t}(f)(x) = -\frac{1}{2t} \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \right) f(x + kv) e^{-|v|/t} dv$$

$$= \int_{-\infty}^{+\infty} f(x-u) \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{k} \cdot \frac{e^{-|u|/(kt)}}{2t} \right] du, t > 0, x \in \mathbb{R}.$$

(v) Starting from the well known Gauss-Weierstrass operator $W_t(f)(x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(x-v) e^{-v^2/t} dv$, we can define its Jackson type generalization by (see, e.g., [8])

$$\begin{aligned} W_{n,t}(f)(x) &= -\frac{1}{2C^*(t)} \cdot \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} f(x+kv) e^{-v^2/t} \right) dv \\ &= \int_{-\infty}^{+\infty} f(x-u) \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{k} \cdot \frac{e^{-u^2/(kt)}}{2C^*(t)} \right] du, t > 0, x \in \mathbb{R}, \end{aligned}$$

where $C^*(t) = \int_0^\infty e^{-u^2/t} du = \frac{\sqrt{t\pi}}{2}$. Therefore, for any $n \in \mathbb{N}$, we can write

$$W_{n,t}(f)(x) = \int_{-\infty}^{+\infty} f(x-u) \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{k} \cdot \frac{e^{-u^2/(kt)}}{\sqrt{\pi t}} \right] du.$$

3 Approximation Properties and Applications to PDE

Concerning the convolution operators defined in Section 2, we can state the following approximation properties and applications to PDE.

Theorem 3.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly continuous on \mathbb{R} .*

(i) *We have*

$$|S_t(f)(x) - f(x)| \leq 4\omega_1(f; t)_{\mathbb{R}}, t > 0, x \in \mathbb{R}.$$

The solution of the initial value problem

$$\frac{\partial u}{\partial t}(x, t) = t^3 \frac{\partial^4 u}{\partial x^4}(x, t) - t^2 \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) + 3t \frac{\partial^2 u}{\partial x^2}(x, t), \quad \lim_{s \searrow 0} u(x, s) = f(x), t > 0, x \in \mathbb{R}$$

is $u(x, t) := S_t(f)(x)$.

(ii) *We have*

$$|P_t(f)(x) - f(x)| \leq 2\omega_1(f; t)_{\mathbb{R}}, t > 0, x \in \mathbb{R}.$$

The solution of the initial value problem

$$\frac{\partial u}{\partial t}(x, t) = t^2 \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) + 2t \frac{\partial^2 u}{\partial x^2}(x, t), \quad \lim_{s \searrow 0} u(x, s) = f(x), t > 0, x \in \mathbb{R}$$

is $u(x, t) := P_t(f)(x)$.

(iii) *We have*

$$|E_t(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{1}{t}\right)_{\mathbb{R}}, t > 0, x \in \mathbb{R}.$$

The solution of the initial value problem

$$\frac{\partial u}{\partial t}(x, t) = \frac{1}{t^2} \cdot \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) - \frac{2}{t^3} \cdot \frac{\partial^2 u}{\partial x^2}(x, t), \quad \lim_{s \rightarrow \infty} u(x, s) = f(x), t > 0, x \in \mathbb{R}$$

is $u(x, t) := E_t(f)(x)$.

(iv) We have

$$|P_{n,t}(f)(x) - f(x)| \leq C_n \cdot \omega_{n+1}(f; t)_{\mathbb{R}}, n \in \mathbb{N}, t > 0, x \in \mathbb{R}$$

and

$$P_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot u_k(x, t),$$

where $u_k(x, t) = P_{kt}(f)(x) = \frac{1}{2kt} \cdot \int_{-\infty}^{+\infty} f(x-u) e^{-|u|/(kt)} du$, $k = 1, \dots, n+1$ are solutions of the initial value problems

$$\frac{\partial u_k}{\partial t}(x, t) = k^2 \frac{\partial^3 u_k}{\partial x^2 \partial t}(x, t) + 2k^2 t \cdot \frac{\partial^2 u_k}{\partial x^2}(x, t), \quad \lim_{s \searrow 0} u_k(x, s) = f(x), t > 0, x \in \mathbb{R}.$$

(v) We have

$$|W_{n,t}(f)(x) - f(x)| \leq C_n \cdot \omega_{n+1}(f; t)_{\mathbb{R}}, n \in \mathbb{N}, t > 0, x \in \mathbb{R}$$

and

$$W_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot u_k(x, t),$$

where $u_k(x, t) = W_{kt}(f)(x) = \frac{1}{k\sqrt{\pi t}} \cdot \int_{-\infty}^{+\infty} f(x-u) e^{-u^2/(kt)} du$, $k = 1, \dots, n+1$ are solutions of the initial value problems

$$\frac{\partial u_k}{\partial t}(x, t) = \frac{k}{4} \cdot \frac{\partial^2 u_k}{\partial x^2}(x, t), \quad \lim_{s \searrow 0} u_k(x, s) = f(x), t > 0, x \in \mathbb{R}.$$

Proof. Since for the convolution operator given by (1), in general we have $d(t, v) \geq 0$, for all $t > 0$ and $v \in \mathbb{R}$, by the standard method we easily get

$$\begin{aligned} |O_t(f)(x) - f(x)| &\leq \int_{-\infty}^{+\infty} |f(x-v) - f(x)| d(t, v) dv \leq \int_{-\infty}^{+\infty} \omega_1(f; |v|)_{\mathbb{R}} d(t, v) dv \\ &\leq 2\omega_1(f; \varphi(t))_{\mathbb{R}}, \end{aligned}$$

where $\omega_1(f; \delta)_{\mathbb{R}} = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, |x - y| \leq \delta\}$ and $\varphi(t) = \int_{-\infty}^{+\infty} |v| \cdot d(t, v) dv$.

Evidently that this method is useful only if $\varphi(t) < +\infty$ for all $t > 0$.

In order to deduce the PDE equations satisfied by various convolution operators, we will need the concepts of Fourier transform of a function g , defined by

$$F(g)(\xi) = \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} g(x) e^{-i\xi x} dx, \quad \text{if } \int_{-\infty}^{+\infty} |g(x)| dx < +\infty,$$

and of inverse Fourier transform defined by

$$F^{-1}(\hat{g})(x) = g(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} \hat{g}(\xi) e^{i\xi x} d\xi.$$

(i) By making the change of variable $v = \sqrt{2}ts$, we get

$$\begin{aligned} \varphi(t) &= \frac{1}{t^3} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty v^3 e^{-v^2/(2t^2)} dv = \frac{1}{t^3} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty (2\sqrt{2}t^3 s^3) e^{-s^2} (\sqrt{2}t) ds \\ &= \frac{4\sqrt{2}}{\sqrt{\pi}} t \int_0^\infty s^3 e^{-s^2} ds = \frac{2\sqrt{2}}{\sqrt{\pi}} t < 2t, \end{aligned}$$

which immediately implies

$$|S_t(f)(x) - f(x)| \leq 4\omega_1(f; t)_{\mathbb{R}}, t > 0, x \in \mathbb{R}.$$

Taking into account the uniform continuity of f , the above inequality immediately implies that $\lim_{t \searrow 0} S_t(f)(x) = f(x)$, for all $x \in \mathbb{R}$. Therefore we may take, by convention, $S_0(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

Now, in order to deduce the PDE satisfied by $S_t(f)(x)$, we write it in the form

$$S_t(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \cdot \frac{(x-y)^2 e^{-(x-y)^2/(2t^2)}}{t^3} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \cdot \hat{g}_t(y-x) dy.$$

Here, by using WolframAlpha program of calculation, we obtain

$$g_t(\xi) = F_w^{-1}[w^2 e^{-w^2/(2t^2)} / t^3](\xi, t) = e^{-t^2 \xi^2/2} (1 - t^2 \xi^2),$$

which implies

$$\begin{aligned} S_t(f)(x) &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} f(y) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(y-x)\xi} e^{-t^2 \xi^2/2} (1 - t^2 \xi^2) d\xi \right] dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-iy\xi} f(y) dy \right] e^{ix\xi} e^{-t^2 \xi^2/2} (1 - t^2 \xi^2) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{ix\xi} \hat{f}(\xi) e^{-t^2 \xi^2/2} (1 - t^2 \xi^2) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{ix\xi} \hat{u}(\xi, t) d\xi := u(x, t), \end{aligned}$$

where

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cdot e^{-t^2 \xi^2/2} (1 - t^2 \xi^2).$$

This is equivalent to $\hat{u}(\xi, t) \cdot \frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} = \hat{f}(\xi)$, which is equivalent to

$$\frac{\partial}{\partial t} \left[\hat{u}(\xi, t) \cdot \frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} \right] = \frac{\partial \hat{u}}{\partial t}(\xi, t) \cdot \frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} + \hat{u}(\xi, t) \cdot \left(\frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} \right)'_t = 0.$$

Note that the above relation can be evidently written under the form

$$\frac{\partial}{\partial t} \left[\hat{u}(\xi, t) \cdot \frac{1}{F_w^{-1}(d(t, w))(\xi, t)} \right] = 0,$$

where $d(t, x)$ is the Maxwell-Boltzmann type probability density in Definition 2.1, (i), entering in the formula for $S_t(f)(x)$.

After simple calculation, the above formula is formally equivalent to (of course for $1 \neq t^2\xi^2$)

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) + t^2 \left(-\xi^2 \cdot \frac{\partial \hat{u}}{\partial t}(\xi, t) \right) - 3t[-\xi^2 \hat{u}(\xi, t)] - t^3 \cdot [\xi^4 \hat{u}(\xi, t)] = 0.$$

Now, taking into account that

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) = \widehat{\frac{\partial u}{\partial t}}(\xi, t), \quad \widehat{\frac{\partial^2 u}{\partial x^2}}(\xi, t) = -\xi^2 \hat{u}(\xi, t), \quad \widehat{\frac{\partial^4 u}{\partial x^4}}(\xi, t) = \xi^4 \hat{u}(\xi, t),$$

and replacing above, we obtain

$$F \left(\frac{\partial u}{\partial t} + t^2 \frac{\partial^3 u}{\partial x^2 \partial t} - 3t \frac{\partial^2 u}{\partial x^2} - t^3 \frac{\partial^4 u}{\partial x^4} \right) (\xi, t) = 0,$$

that is

$$\frac{\partial u}{\partial t}(x, t) = t^3 \frac{\partial^4 u}{\partial x^4}(x, t) - t^2 \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) + 3t \frac{\partial^2 u}{\partial x^2}(x, t).$$

Finally, following the above steps in inverse order, we arrive at the conclusion in the statement.

(ii) In fact, in [2], p. 142, Corollary 3.4.2, it was obtained more than in the statement :

$$|f(x) - P_t(f)(x)| \leq C\omega_2(f; t)_{\mathbb{R}}.$$

But, for the reader's convenience, we apply here the standard reasonings described just before the proof of (i).

We obtain :

$$|f(x) - P_t(f)(x)| \leq 2\omega_1(f; \varphi(t))_{\mathbb{R}},$$

where

$$\varphi(t) = \frac{1}{2t} \int_{-\infty}^{+\infty} |v| e^{-|v|/t} dv = \frac{1}{t} \int_0^{+\infty} v e^{-v/t} dv.$$

By making the change of variable $v/t = u$, it easily follows that $\varphi(t) = t$, which implies

$$|f(x) - P_t(f)(x)| \leq 2\omega_1(f; t)_{\mathbb{R}}.$$

In order to deduce the PDE satisfied by $P_t(f)(x)$, we reason exactly as in the above case (i). Indeed, by making use of WolframAlpha program, we get

$$F_w^{-1}[e^{-|w|/t}/(2t)](\xi, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1 + t^2\xi^2}$$

and similar reasonings with those in the case (i), immediately leads to

$$P_t(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi} \hat{u}(\xi, t) d\xi := u(x, t),$$

where

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cdot \frac{1}{t^2\xi^2 + 1}.$$

In fact, directly as in the proof of Theorem 3.1, (i), we can write

$$\frac{\partial}{\partial t} \left[\hat{u}(\xi, t) \cdot \frac{1}{F_w^{-1}(d(t, w))(\xi, t)} \right] = 0,$$

where $d(t, x)$ is the Laplace type probability density in Definition 2.1, (ii), entering in the formula for $P_t(f)(x)$.

Therefore,

$$\frac{\partial}{\partial t} [\hat{u}(\xi, t) \cdot (1 + t^2\xi^2)] = \frac{\partial \hat{u}}{\partial t}(\xi, t) + t^2\xi^2 \frac{\partial \hat{u}}{\partial t}(\xi, t) + 2t\xi^2 \hat{u}(\xi, t) = 0,$$

which immediately leads to

$$\frac{\partial u}{\partial t}(x, t) = t^2 \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) + 2t \frac{\partial^2 u}{\partial x^2}(x, t).$$

Following the above steps, now from the end to the beginning, we arrive at the conclusion in the statement.

(iii) Firstly, we observe that $E_t(f)(x) = P_{1/t}(f)(x)$, for all $t > 0$ and $x \in \mathbb{R}$. The, by (ii) we immediately get

$$|E_t(f)(x) - f(x)| = |P_{1/t}(f)(x) - f(x)| \leq 2\omega_1 \left(f; \frac{1}{t} \right)_{\mathbb{R}}.$$

Then, according to WolframAlpha, we have $F_w^{-1}(e^{-|w|/t})(\xi, t) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{t}{t^2 + \xi^2}$. This immediately implies

$$F^{-1}(d(t, w))(\xi, t) = \frac{t}{2} \cdot F_w^{-1}(e^{-|w|/t})(\xi, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{t^2}{t^2 + \xi^2}.$$

It follows $\frac{1}{F^{-1}(d(t, w))(\xi, t)} = \sqrt{2\pi} \cdot \frac{t^2 + \xi^2}{t^2} = \sqrt{2\pi} \left(1 + \frac{\xi^2}{t^2} \right)$.

Therefore, denoting $u(x, t) = E_t(f)(x)$, by the method used at the above points, we arrive at the PDE

$$\frac{\partial}{\partial t} \left(\hat{u}(\xi, t) \cdot \left(1 + \frac{\xi^2}{t^2} \right) \right) = \frac{\hat{u}}{\partial t}(\xi, t) \left(1 + \frac{\xi^2}{t^2} \right) + \hat{u}(\xi, t) \left(-\frac{2\xi^2}{t^3} \right) = 0.$$

This immediately leads to the following PDE, satisfied by $u(x, t) = E_t(f)(x)$

$$\frac{\partial u}{\partial t}(x, t) = \frac{1}{t^2} \cdot \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) - \frac{2}{t^3} \cdot \frac{\partial^2 u}{\partial x^2}(x, t).$$

Since $E_t(f)(x) = P_{1/t}(f)(x)$, it follows that $\lim_{t \nearrow \infty} E_t(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

Following the above steps in inverse order, we arrive at the conclusion in the statement.

(iv) Concerning the approximation properties of $P_{n,t}(f)(x)$, in [8] it was obtained the estimate

$$|f(x) - P_{n,t}(f)(x)| \leq \sum_{k=1}^{n+1} k! \cdot \binom{n+1}{k} \cdot \omega_{n+1}(f; t)_{\mathbb{R}},$$

where $\omega_{n+1}(f; \delta) = \sup_{0 \leq h \leq \delta} \{ |\Delta_h^{n+1} f(x); x \in \mathbb{R} \}$, with $\Delta_h^{n+1} = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x + jh)$. This immediately implies $\lim_{t \searrow 0} P_{n,t}(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

In order to deduce the PDE satisfied by $P_{n,t}(f)(x)$, since F_w^{-1} is linear operator and since (by WolframAlpha software) $F_w^{-1}(e^{-|w|/t})(\xi, t) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{t}{t^2 \xi^2 + 1}$, replacing here t by kt , we easily obtain

$$\begin{aligned} F_w^{-1} \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{2tk} e^{-|w|/(kt)} \right] (\xi, t) &= \frac{1}{\sqrt{2\pi}} \cdot \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{kt} \cdot \frac{kt}{k^2 t^2 \xi^2 + 1} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{k^2 t^2 \xi^2 + 1}. \end{aligned}$$

Therefore, denoting $u(x, t) := P_{n,t}(f)(x)$ we immediately get the differential equation

$$\frac{\partial}{\partial t} \left[\hat{u}(\xi, t) \cdot \frac{1}{\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{k^2 t^2 \xi^2 + 1}} \right] = 0,$$

which is equivalent to

$$\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \left[\frac{\partial \hat{u}}{\partial t}(\xi, t) \cdot \frac{1}{k^2 t^2 \xi^2 + 1} + \hat{u}(\xi, t) \cdot \frac{2k^2 t \xi^2}{(k^2 t^2 \xi^2 + 1)^2} \right] = 0.$$

It is worth noting that denoting

$$u_k(x, t) = P_{kt}(f)(x) = \frac{1}{2kt} \cdot \int_{-\infty}^{+\infty} f(x - u) e^{-|u|/(kt)} du,$$

we can write

$$P_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot u_k(x, t),$$

where reasoning as above for $P_t(f)(x)$, we easily obtain

$$\frac{\partial \hat{u}_k}{\partial t}(\xi, t) = k^2 \frac{\partial^3 \hat{u}_k}{\partial x^2 \partial t}(\xi, t) + 2k^2 t \cdot \frac{\partial^2 \hat{u}_k}{\partial x^2}(\xi, t)$$

and which implies

$$\frac{\partial u_k}{\partial t}(x, t) = k^2 \frac{\partial^3 u_k}{\partial x^2 \partial t}(x, t) + 2k^2 t \cdot \frac{\partial^2 u_k}{\partial x^2}(x, t), \quad x \in \mathbb{R}, t > 0, k = 1, \dots, n+1,$$

with $u_k(x, 0) = f(x)$, for all $x \in \mathbb{R}$, $k = 1, \dots, n+1$.

(v) Concerning the approximation properties of $W_{n,t}(f)(x)$, reasoning as in [8], we get the estimate

$$|f(x) - W_{n,t}(f)(x)| \leq C_n \cdot \omega_{n+1}(f; \sqrt{t})_{\mathbb{R}},$$

where $C_n > 0$ is a constant independent of f , t and x . This immediately implies that $\lim_{t \searrow 0} W_{n,t}(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

Now, in order to deduce the PDE satisfied by $W_{n,t}(f)(x)$, since F_w^{-1} is linear operator and since (by WolframAlpha software) $F_w^{-1}(e^{-w^2/t})(\xi, t) = \frac{\sqrt{t} \cdot e^{-t\xi^2/4}}{\sqrt{2}}$, replacing here t by kt , we easily obtain $F_w^{-1}(e^{-w^2/(kt)})(\xi, t) = \frac{\sqrt{kte^{-kt\xi^2/4}}}{\sqrt{2}}$ and

$$\begin{aligned} F_w^{-1} \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{\sqrt{\pi t k}} e^{-w^2/(kt)} \right] (\xi, t) &= \frac{1}{\sqrt{\pi}} \cdot \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{k\sqrt{t}} \cdot \frac{\sqrt{kte^{-kt\xi^2/4}}}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{\sqrt{k}} \cdot e^{-kt\xi^2/4}. \end{aligned}$$

Therefore, denoting $u(x, t) := W_{n,t}(f)(x)$ we immediately get the differential equation

$$\frac{\partial}{\partial t} \left[\hat{u}(\xi, t) \cdot \frac{1}{\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{\sqrt{k}} \cdot e^{-kt\xi^2/4}} \right] = 0,$$

which is equivalent to

$$\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \frac{1}{\sqrt{k}} \left[\frac{\partial \hat{u}}{\partial t}(\xi, t) \cdot e^{-kt\xi^2/4} + \hat{u}(\xi, t) \cdot \frac{k\xi^2}{4} \cdot e^{-kt\xi^2/4} \right] = 0.$$

It is worth noting that denoting

$$u_k(x, t) = W_{kt}(f)(x) = \frac{1}{k\sqrt{\pi t}} \cdot \int_{-\infty}^{+\infty} f(x-u) e^{-u^2/(kt)} du,$$

we can write

$$W_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot u_k(x, t),$$

where reasoning as above but now for $W_t(f)(x)$, we easily obtain

$$\frac{\partial \widehat{u}_k}{\partial t}(\xi, t) = \frac{k}{4} \cdot \frac{\partial^2 \widehat{u}_k}{\partial x^2}(\xi, t)$$

and which implies

$$\frac{\partial u_k}{\partial t}(x, t) = \frac{k}{4} \cdot \frac{\partial^2 u_k}{\partial x^2}(x, t), \quad x \in \mathbb{R}, t > 0, k = 1, \dots, n+1,$$

with $u_k(x, 0) = f(x)$, for all $x \in \mathbb{R}$, $k = 1, \dots, n+1$. □

4 Horseshoe Convolution Operator

In this section we make a separate study for the convolution operator based on a special unbounded density of probability with vertical asymptote at $x = 0$, called "horseshoe density", defined by (see, e.g., [3], [4])

$$d(t, x) = \ln \left(1 + \frac{t^2}{x^2} \right), \quad t > 0, x \in \mathbb{R}.$$

In this case, the real convolution singular integral is an improper integral in the sense of Cauchy principal value and it is defined by

$$\begin{aligned} H_t(f)(x) &= \frac{1}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} f(s) \ln \left(1 + \frac{t^2}{(s-x)^2} \right) ds \\ &= \frac{1}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} f(x-v) \ln \left(1 + \frac{t^2}{v^2} \right) dv, \end{aligned}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is supposed to be such that the principal value (denoted by (p.v.)) of the integral (that is the value $V = \lim_{\varepsilon \searrow 0} \int_{-\infty}^{-\varepsilon} + \lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{+\infty}$) exists finite.

For example, $H_t(f)(x)$ exists finite for any bounded f . Indeed, supposing that $|f(s)| \leq M$ for all $s \in \mathbb{R}$, then for all $x \in \mathbb{R}$ it follows

$$\begin{aligned} |H_t(f)(x)| &\leq \frac{1}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} |f(s)| \ln \left(1 + \frac{t^2}{(s-x)^2} \right) ds \\ &\leq M \cdot \frac{1}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} \ln \left(1 + \frac{t^2}{(s-x)^2} \right) ds = M. \end{aligned}$$

Indeed, taking into account that $\int \ln \left(1 + \frac{1}{x^2} \right) dx = x \ln \left(1 + \frac{1}{x^2} \right) - 2 \cot^{-1}(x) + C$ (C a constant), by the change of variable $s = (x-v)/t$, we easily obtain

$$\int_{\varepsilon}^{+\infty} \ln \left(1 + \frac{t^2}{(x-v)^2} \right) dv = t \int_{-\infty}^{(x-\varepsilon)/t} \ln \left(1 + \frac{1}{s^2} \right) ds$$

$$= t \left[\frac{x - \varepsilon}{t} \ln \left(1 + \frac{t^2}{(x - \varepsilon)^2} \right) - 2 \cot^{-1}((x - \varepsilon)/t) + 2\pi \right].$$

Passing to limit with $\varepsilon \searrow 0$, we obtain

$$\lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{+\infty} \ln \left(1 + \frac{t^2}{(x - v)^2} \right) dv = t \cdot \left[\frac{x}{t} \cdot \ln \left(1 + \frac{t^2}{x^2} \right) - 2 \cot^{-1}(x/t) + 2\pi \right].$$

By similar calculation, we obtain

$$\begin{aligned} \int_{-\infty}^{-\varepsilon} \ln \left(1 + \frac{t^2}{(x - v)^2} \right) dv &= t \int_{(x+\varepsilon)/t}^{+\infty} \ln \left(1 + \frac{1}{t^2} \right) dt \\ &= t \left[-\frac{x + \varepsilon}{t} \ln \left(1 + \frac{t^2}{(x + \varepsilon)^2} \right) + 2 \cot^{-1}((x + \varepsilon)/t) \right]. \end{aligned}$$

Passing to limit with $\varepsilon \searrow 0$, we obtain

$$\lim_{\varepsilon \searrow 0} \int_{-\infty}^{-\varepsilon} \ln \left(1 + \frac{t^2}{(x - v)^2} \right) dv = t \cdot \left[-\frac{x}{t} \cdot \ln \left(1 + \frac{t^2}{x^2} \right) + 2 \cot^{-1}(x/t) \right].$$

Therefore, it follows

$$(p.v.) \int_{-\infty}^{+\infty} \ln \left(1 + \frac{t^2}{(s - x)^2} \right) ds = 2\pi t,$$

which finally leads to the required inequality.

The first main result of this section is the following.

Theorem 4.1. *Let $\alpha \in (0, 1)$.*

(i) *Let us suppose that f belongs to the class denoted by $\text{Lip } \alpha$ and consisting in the class of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that there exists a constant $C > 0$ (depending on f), with the property that $|f(x) - f(y)| \leq C|x - y|^\alpha$, for all $x, y \in \mathbb{R}$. For all $x \in \mathbb{R}$ and $t > 0$ we have*

$$|H_t(f)(x) - f(x)| \leq C_{\alpha, f} \cdot t^{\alpha/2},$$

where $C_{\alpha, f} > 0$ depends on α and f but is independent of t .

(ii) *Let $1 \leq p < +\infty$ and suppose that $f \in L^p(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R}; \int_{\mathbb{R}} |f(x)|^p dx < +\infty\}$ is such that*

$$\int_{-\infty}^{+\infty} |f(x + h) - f(x)|^p dx \leq c_{p, f} \cdot |h|^\alpha, \text{ for all } h \in \mathbb{R}.$$

Then, for all $t > 0$ we have

$$\|H_t(f) - f\|_p^p \leq C_{f, \alpha, p} \cdot t^{\alpha/2},$$

where $\|f\|_p = (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$ and $C_{f, \alpha, p} > 0$ depends on f, α, p but is independent of t .

Proof. (i) Since by the calculation made just before the statement we have $H_t(1)(x) = 1$, it immediately follows

$$\begin{aligned}
|H_t(f)(x) - f(x)| &= \left| \frac{1}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} (f(x-v) - f(x)) \ln \left(1 + \frac{t^2}{v^2} \right) dv \right| \\
&\leq \frac{1}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} |f(x-v) - f(x)| \ln \left(1 + \frac{t^2}{v^2} \right) dv \\
&\leq \frac{C}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} |v|^\alpha \ln \left(1 + \frac{t^2}{v^2} \right) dv \\
&= \frac{C}{\pi t} \cdot (p.v.) \int_0^{+\infty} v^\alpha \ln \left(1 + \frac{t^2}{v^2} \right) dv \\
&= \frac{C}{\pi} \cdot t^{\alpha/2} \cdot (p.v.) \int_0^{+\infty} s^\alpha \ln \left(1 + \frac{1}{s^2} \right) ds.
\end{aligned}$$

But integrating by parts and then applying the L'Hopital's rule, we easily get

$$(p.v.) \int_0^{+\infty} s^\alpha \ln \left(1 + \frac{1}{s^2} \right) ds = \frac{2}{\alpha + 1} \cdot \int_0^{+\infty} \frac{s^\alpha}{1 + s^2} ds,$$

where it is easy to show that for $0 < \alpha < 1$ we have $\int_0^{+\infty} \frac{s^\alpha}{1+s^2} ds < +\infty$.

This implies the required estimate.

(ii) We will use the following well-known Jensen type inequality for integrals : if $a, b \in \overline{\mathbb{R}}$, $a < b$, $\int_a^b G(u) du = 1$, $G(u) \geq 0$ for all $u \in (a, b)$ and $\Phi(s)$ is convex function over the range of the measurable function of real variable F , then

$$\varphi \left(\int_a^b F(u) G(u) du \right) \leq \int_a^b \varphi(F(u)) G(u) du.$$

Also, the inequalities $(a+b)^p \leq 2^{p-1}(a^p + b^p) \leq 2^p(a+b)^p$ valid for all $a, b \geq 0$, $1 \leq p < +\infty$, will be useful.

We get

$$|H_t(f)(x) - f(x)| \leq \frac{1}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} |f(x-v) - f(x)| \ln \left(1 + \frac{t^2}{v^2} \right) dv,$$

which implies

$$\begin{aligned}
|H_t(f)(x) - f(x)|^p &\leq \left(\frac{1}{2\pi t} \cdot (p.v.) \int_{-\infty}^{+\infty} |f(x-v) - f(x)| \ln \left(1 + \frac{t^2}{v^2} \right) dv \right)^p \\
&= \lim_{\varepsilon \searrow 0} \left[\frac{1}{2\pi t} \int_{-\infty}^{-\varepsilon} |f(x-v) - f(x)| \ln \left(1 + \frac{t^2}{nv^2} \right) dv \right. \\
&\quad \left. + \frac{1}{2\pi t} \int_{\varepsilon}^{+\infty} |f(x-v) - f(x)| \ln \left(1 + \frac{t^2}{v^2} \right) dv \right]^p
\end{aligned}$$

$$\begin{aligned} &\leq 2^{p-1} \lim_{\varepsilon \searrow 0} \left(\frac{1}{2\pi t} \int_{-\infty}^{-\varepsilon} |f(x-v) - f(x)| \ln \left(1 + \frac{t^2}{v^2} \right) dv \right)^p \\ &+ 2^{p-1} \lim_{\varepsilon \searrow 0} \left(\frac{1}{2\pi t} \int_{\varepsilon}^{+\infty} |f(x-v) - f(x)| \ln \left(1 + \frac{t^2}{v^2} \right) dv \right)^p. \end{aligned}$$

Now, taking into account the calculations in this section made just before the statement of Theorem 2.1 and denoting

$$\begin{aligned} 0 < A_{t,x,\varepsilon} &:= \frac{1}{2\pi t} \int_{-\infty}^{-\varepsilon} \ln \left(1 + \frac{t^2}{(x-v)^2} \right) dv \\ &= \frac{1}{2\pi} \left[-\frac{x+\varepsilon}{t} \ln \left(1 + \frac{t^2}{(x+\varepsilon)^2} \right) + 2 \cot^{-1} \left(\frac{x+\varepsilon}{t} \right) \right] < +\infty \end{aligned}$$

and

$$\begin{aligned} B_{t,x,\varepsilon} &:= \frac{1}{2\pi t} \int_{\varepsilon}^{+\infty} \ln \left(1 + \frac{t^2}{(x-v)^2} \right) dv \\ &= \frac{1}{2\pi} \left[\frac{x-\varepsilon}{t} \ln \left(1 + \frac{t^2}{(x-\varepsilon)^2} \right) - 2 \cot^{-1} \left(\frac{x-\varepsilon}{t} \right) + 2\pi \right], \end{aligned}$$

it follows

$$\int_{-\infty}^{-\varepsilon} \left[\frac{1}{2\pi t} \cdot \frac{1}{A_{t,x,\varepsilon}} \ln \left(1 + \frac{t^2}{(x-v)^2} \right) \right] dv = 1$$

and

$$\int_{\varepsilon}^{+\infty} \left[\frac{1}{2\pi t} \cdot \frac{1}{B_{t,x,\varepsilon}} \ln \left(1 + \frac{t^2}{(x-v)^2} \right) \right] dv = 1.$$

Replacing in the above inequality and applying the Jensen's inequality for $\Phi(s) = s^p$, we obtain

$$\begin{aligned} &|H_t(f)(x) - f(x)|^p \\ &\leq 2^{p-1} \lim_{\varepsilon \searrow 0} \left(A_{t,x,\varepsilon} \cdot \int_{-\infty}^{-\varepsilon} |f(x-v) - f(x)| \left[\frac{1}{2\pi t} \cdot \frac{1}{A_{t,x,\varepsilon}} \ln \left(1 + \frac{t^2}{v^2} \right) \right] dv \right)^p \\ &+ 2^{p-1} \lim_{\varepsilon \searrow 0} \left(B_{t,x,\varepsilon} \cdot \int_{\varepsilon}^{+\infty} |f(x-v) - f(x)| \left[\frac{1}{2\pi t} \cdot \frac{1}{B_{t,x,\varepsilon}} \ln \left(1 + \frac{t^2}{v^2} \right) \right] dv \right)^p \\ &\leq 2^{p-1} \lim_{\varepsilon \searrow 0} \left(A_{t,x,\varepsilon} \cdot \int_{-\infty}^{-\varepsilon} |f(x-v) - f(x)|^p \left[\frac{1}{2\pi t} \cdot \frac{1}{A_{t,x,\varepsilon}} \ln \left(1 + \frac{t^2}{v^2} \right) \right] dv \right) \\ &+ 2^{p-1} \lim_{\varepsilon \searrow 0} \left(B_{t,x,\varepsilon} \cdot \int_{\varepsilon}^{+\infty} |f(x-v) - f(x)|^p \left[\frac{1}{2\pi t} \cdot \frac{1}{B_{t,x,\varepsilon}} \ln \left(1 + \frac{t^2}{v^2} \right) \right] dv \right). \end{aligned}$$

Simplifying inside of the parenthesis with $A_{t,x,\varepsilon}$ and $B_{t,x,\varepsilon}$, integrating the above inequality with respect to $x \in \mathbb{R}$ and applying the Fubini's theorem, we easily obtain

$$\|H_t(f) - f\|_p^p \leq 2^{p-1} \cdot (p.v.) \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |f(x-v) - f(x)|^p dx \right] \frac{1}{2\pi t} \ln \left(1 + \frac{t^2}{v^2} \right) dv$$

$$\begin{aligned}
&\leq c_{f,p} \cdot 2^{p-1}(p.v.) \int_{-\infty}^{+\infty} |v|^\alpha \frac{1}{2\pi t} \ln \left(1 + \frac{t^2}{v^2} \right) dv = c_{f,p} \cdot 2^p(p.v.) \int_0^{+\infty} v^\alpha \frac{1}{2\pi t} \ln \left(1 + \frac{t^2}{v^2} \right) dv \\
&\leq 2^p \cdot C_{f,p,\alpha} \cdot t^{\alpha/2},
\end{aligned}$$

which implies the estimate in the statement. \square

Remark 4.2. If $\alpha = 1$, then the reasonings for the statement of Theorem 4.1, (i), do not work. Indeed, in this case we get

$$\begin{aligned}
(p.v.) \int_0^{+\infty} v \ln \left(1 + \frac{t^2}{v^2} \right) dv &= t^2 \cdot (p.v.) \int_0^{+\infty} s \ln \left(1 + \frac{1}{s^2} \right) ds \\
&= t^2 \left[\frac{1}{2} \cdot x^2 \cdot \ln \left(1 + \frac{1}{x^2} \right) + \frac{1}{2} \cdot \ln(1 + x^2) \right]_0^{+\infty} = +\infty.
\end{aligned}$$

Remark 4.3. It is natural to ask what partial differential equation would satisfy $H_t(f)(x) = (p.v.) \int_{-\infty}^{+\infty} f(x-v) \cdot d(t,v)dv$, where $d(t,x) = \frac{1}{2t\pi} \ln \left(1 + \frac{t^2}{x^2} \right)$.

Here, by Theorem 4.1, (i), we have that $\lim_{t \searrow 0} H_t(f)(x) = f(x)$ for all $x \in \mathbb{R}$ and for f Lipschitz α function with $0 < \alpha < 1$.

In this sense, we have :

Theorem 4.4. Let f be of Lipschitz class $\alpha \in (0, 1)$ on \mathbb{R} , with f' uniformly continuous on \mathbb{R} . Then $u(x, t) = H_t(f)(x)$ is solution of the initial value problem

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial u}{\partial x}(x, t) - \frac{1}{t}u(x, t), \quad \lim_{s \searrow 0} u(x, s) = f(x), t > 0, x \in \mathbb{R}.$$

Proof. Denoting $u(x, t) = \frac{1}{2\pi t}(p.v.) \int_{-\infty}^{+\infty} f(x-v) \cdot \ln \left(1 + \frac{t^2}{v^2} \right) dv$, we formally get

$$\begin{aligned}
\frac{\partial u}{\partial t}(x, t) &= \frac{1}{2\pi} (p.v.) \int_{-\infty}^{+\infty} f(x-v) \left(\frac{\ln(1 + t^2/v^2)}{t} \right)'_t dv \\
&= \frac{1}{2\pi t} \int_{-\infty}^{+\infty} f(x-v) \cdot \frac{2t}{v^2 + t^2} dv - \frac{1}{t} \cdot \frac{1}{2\pi t} (p.v.) \int_{-\infty}^{+\infty} f(x-v) \ln(1 + t^2/v^2) dv \\
&= \frac{1}{2\pi t} \int_{-\infty}^{+\infty} f(x-v) \cdot d[\ln(1 + t^2/v^2)] - \frac{1}{t}u(x, t) \\
&= \frac{1}{2\pi t} \cdot f(x-v) \cdot \ln(1 + t^2/v^2)|_{-\infty}^{+\infty} + \frac{1}{2\pi t} \int_{-\infty}^{+\infty} f'(x-v) \ln(1 + t^2/v^2) dv - \frac{1}{t}u(x, t) \\
&= \frac{1}{2\pi t} \cdot f(x-v) \cdot \ln(1 + t^2/v^2)|_{-\infty}^{+\infty} + \frac{\partial u}{\partial x}(x, t) - \frac{1}{t}u(x, t).
\end{aligned}$$

Since by Theorem 4.1, (i), f is uniformly continuous on \mathbb{R} , it follows that (see, e.g., [5], p. 48, Problème 4 or [6]) f is of linear growth, that is $|f(x)| \leq \alpha|x| + \beta$, for all $x \in \mathbb{R}$, with $\alpha, \beta \geq 0$. This immediately implies that $\frac{1}{2\pi t} \cdot f(x-v) \cdot \ln(1 + t^2/v^2)|_{-\infty}^{+\infty} = 0$, which leads to the fact that $u(x, t) = H_t(f)(x)$ is solution of the initial value problem in the statement. \square

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